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Partition function for a nonlinear supersymmetric model

R Mendoza†, J Rojas‡ and F Moraes§

† Universidade Federal de Pernambuco, CCEN-DMAT, Recife-PE, Brazil
‡ Universidade Federal da Paraíba, CCEN-DMAT, J Pessoa-PB, Brazil
§ Universidade Federal de Pernambuco, CCEN-LFTC, Recife-PE, Brazil

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Abstract. We analyse the nonlinear supersymmetric model associated with the Polyakov Euclidean σ-model. In this theory the main point is to be able to compute the partition and correlation functions of interesting statistical variables. We briefly review some tools for renormalizing the partition function and we carry out the computations of the n-instanton contributions for a toy model on S^1; these are given explicitly in terms of specific hypergeometric functions.

1. Introduction

The main objective of this paper is to give a self-contained justification of the essential tools used for the renormalization of the partition function and the explicit computation of the n-instanton contribution to the partition function for a toy model on S^1. This contribution is given in terms of specific hypergeometric functions.

In relation to the general ideas and physical background we strongly recommend the papers of Witten and Vafa at the ICM 1986 and 1998 respectively [1, 2].

In section 2 we review some concepts of statistical mechanical and the renormalization tools needed. In section 3 we define the Dirac operator and the *super-energy*. In section 4 we perform the calculation for our toy model and finally in section 5 we present our conclusions.

2. Notations and preliminaries

2.1. Basic statistical mechanics

In this section we review briefly the basic concepts of statistical mechanics [3], semiclassical approximation, renormalization theory and supersymmetry [4] required for defining a nonlinear supersymmetric model.

Definition 2.1. *Let $\mathcal{S}$ be a set and $E : \mathcal{S} \longrightarrow \mathbb{R}$ a function. The pair $(\mathcal{S}, E)$ will be called a physical system. The elements of $\mathcal{S}$ are denominated configurations or states of the system and E will be the energy function.*

Definition 2.2. *The partition function Z associated with the physical system $(\mathcal{S}, E)$ is given by*

$$Z = \int_{\mathcal{S}} \exp(-\beta E)\,\mathrm{d}\mu \tag{1}$$

where β is related to the temperature T through the formula $\beta^{-1} = kT$ and k is the Boltzmann constant. μ denotes a positive measure over $\mathcal{S}$.

Definition 2.3. *Let* $(\mathcal{S}, E)$ *be a physical system. A measurable real-valued function* X *on* $\mathcal{S}$ *is called a physical observable and its mean value corresponds to*

$$\langle X\rangle = \frac{\int_{\mathcal{S}} \exp(-\beta E) X \, \mathrm{d}\mu}{Z}. \tag{2}$$

We will introduce the renormalization and semiclassical approximation techniques needed to compute a finite value for the partition function Z.

The renormalization technique is related to an isometric action A of a Lie group G on the Riemannian manifold $\mathcal{S}$. We denote by $\mathcal{G}$ the Riemannian metric on $\mathcal{S}$ and by h a bi-invariant metric on G [5]. In [6] the renormalized partition function Z_{R} is defined by

$$Z_{\mathrm{R}} = \frac{Z}{\int_G \nu_h} = \int_{\mathcal{S}/G} \exp(-\beta E) \mathrm{FP} \frac{1}{\mathrm{vol}(K_{\bar{s}})} \nu_{\overline{\mathcal{G}}}. \tag{3}$$

Here ν_h is the volume form of h, $\overline{\mathcal{G}}$ is the metric induced in the orbit space by the metric $\mathcal{G}$ of $\mathcal{S}$ and $\nu_{\overline{\mathcal{G}}}$ the volume form on $\mathcal{S}/G$. K_s is the stabilizer of s and $\mathrm{vol}(K_{\bar{s}})$ denotes the volume of K_s with the volume induced by h on any closed subgroup of G. FP denotes the Faddeev–Popov determinant defined by

$$\mathrm{FP}(\bar{s}) = \sqrt{\det A_s'^* \circ A_s'}. \tag{4}$$

Here A_s' is the derivative of $g \longmapsto s \cdot g$ at the identity of G.

Another important tool that will help us to compute Z is the semiclassical approximation. In fact, this corresponds to

$$Z^{\mathrm{SC}} = \int_{\mathcal{PC}(E)} \exp(-\beta E) \frac{1}{\sqrt{\det \beta \mathcal{H}^E}} \nu_{\tilde{\mathcal{G}}} \tag{5}$$

where $\mathcal{H}^E$ denotes the operator associated with the Hessian of E, restricted to the orthogonal of the tangent space of $\mathcal{PC}(E)$, according to the metric induced by $\mathcal{G}$ ($\mathcal{PC}(E)$ is the set of critical points of E in $\mathcal{S}$). $\tilde{\mathcal{G}}$ denotes the restriction of the metric $\mathcal{G}$ to the sub-manifold $\mathcal{PC}(E)$.

3. The nonlinear supersymmetric model

In this section we will briefly discuss the nonlinear supersymmetric model for arbitrary source Σ and target X. Supersymmetric means here, precisely, that we consider fermions ψ as elements of a Grassmann fibre bundle over the Cartesian product of the space of all Riemannian metrics g on the manifold Σ and over the space of all infinitely differentiable maps ϕ from Σ to X. We are aware that supersymmetry is used when the tangent bundle of X admits isometries for the given metric h that endow each tangent spaces with complex, quaternionic, octonionic structure, but nevertheless we retain the same word because the inspiration for this work originates from the aim of understanding supersymmetry. The first field g describes gravitation in the Euclidean regime and the second ϕ bosonic states. In this paper we will not discuss the representation of bosonic states as connections of a principal bundle over Σ.

We begin with two differentiable manifolds Σ and X. Let us denote by $\mathcal{M} = \mathcal{M}(\Sigma)$ the space of all Riemannian metrics on Σ and by $\mathcal{C}^\infty(\Sigma, X)$ the space of all infinitely differentiable maps between them. We denote by $\mathcal{B}$ the Cartesian product of both spaces. The group of preserving orientation diffeomorphism of Σ will be designated by $\mathcal{D}$.

We proceed to describe the Grassmannian bundle over $\mathcal{B}$.

Let us fix a Riemannian metric h on X. For a given pair (g, ϕ) we will introduce the Dirac operator $\mathbb{D}_{(g,\phi)}$ as follows. First of all we will define a connection on the Clifford bundle $\mathcal{C}_g$ over Σ from the metric g and on $\phi^* T(X)$ the pullback connection associated with h via the map ϕ. The desired connection on $\mathcal{C}_g \otimes \phi^* T(X)$ is obtained by tensoring both connections. Let us denote by ∇ the connection so obtained for $\mathcal{C}_g \otimes \phi^* T(X)$, and E_i with index i running from one up to the dimension of Σ, an orthonormal oriented frame for $T(\Sigma)$.

Definition 3.1. *The Dirac operator associated with the metric g and to the map ϕ is given by*

$$\mathbb{D}_{(g,\phi)} = \Sigma_{i=1}^{m} E^i \nabla_{E_i}. \tag{6}$$

In the above formula m denotes the dimension of Σ.

Remark 3.2. The Dirac operator is a first-order formally self-adjoint differential operator; its spectrum is real unbounded below and above. Its positive (negative) eigenvalues will be denoted by λ_i^+ (λ_i^-) respectively.

We proceed now to describe the physical system. We denote by $\mathcal{B}$ the product space of $\mathcal{M}(\Sigma)$ by $\mathcal{C}^\infty(\Sigma, X)$. Let us consider the bundle of Grassmann algebras over $\mathcal{B}$: for each (g, ϕ) the fibre over it will be the Grassmann algebra generated by one orthonormal basis of eigensections of the corresponding Dirac operator. The space of states, $\mathcal{S}$ will be the disjoint union of all the fibres described above: from now on it will be denoted by $\mathcal{E}$. We need to generalize the notion of the energy function given at the beginning in order to take into account the fermionic nature of the elements of the fibre. In fact the *super-energy* $\hat{E}$ will be a section of the bundle $\mathcal{E} \otimes \mathcal{E}^*$ defined as follows:

$$\hat{E}(\psi) = \alpha \int s(g)\, \nu_g + \tfrac{1}{2}\beta \int \mathrm{Tr}_g\, \phi^* h \nu_g + \gamma \mathbb{D}_{(g,\phi)} \psi \cdot \psi + \delta \int \phi^* B. \tag{7}$$

We will briefly comment on each term of the sum above.

The first term, E_g, is the total energy associated with the gravitational field g. In fact, $s(g)$ coincides, for a four-dimensional Σ, with the energy density of the Lorentzian metric that determines the light path in spacetime (the geodesics of the given g) [7]. On the mathematical side, $s(g)$ is the scalar curvature of g.

The second term, E_{b}^g, is the bosonic energy of the field ϕ coupled with the gravitational field g. For a one-dimensional Σ this should be related to the kinetic energy of a point particle.

The third term, the fermionic energy, is given by the following explicit formal expression:

$$\mathbb{D}_{(g,\phi)} \psi \cdot \psi = \sum \lambda_i^+ \overline{\psi_i^+} \cdot \psi_i^+ + \lambda_i^- \overline{\psi_i^-} \cdot \psi_i^-. \tag{8}$$

The last term, E_{t}, is the topological energy, which depends on the homotopy class of ϕ and on the De Rham cohomology class of B.

The coupling constants α, β, γ and δ are introduced by dimensional reasons and also in order to reflect the force of the interactions between the different fields of the model.

We will define the partition function associated with the super-energy $\hat{E}$ by the following expression:

$$Z_{\mathrm{R}}^{\mathrm{SC}} = \int_{\mathcal{M}/\mathcal{D}} \exp(-\alpha E_g) \left[\int_{\mathcal{PC}(E_{\mathrm{b}}^g)} \exp(-\delta E_{\mathrm{t}}) \exp(-\beta E_{\mathrm{b}}^g) \frac{\det(\gamma \mathbb{D}_{(g,\phi)})}{\sqrt{\det \beta \mathcal{H}^E}} \nu_{\tilde{\mathcal{G}}} \right] \frac{\mathrm{FP}(\overline{g})}{\mathrm{vol}(K_{\overline{g}})} \nu_{\overline{\mathcal{G}}}. \tag{9}$$

In order to restore the important physical constants kT in formula (9) and in what follows, it is sufficient to substitute each coupling constant simultaneously, let us say α by $\frac{\alpha}{kT}$ etc.

4. Partition function for a model on S^1

In this section we will compute the n-instanton contribution to the partition function for a system of gravitational fields, bosons and fermions on $\Sigma = S^1$. We also choose X as a circumference of radius one with h the unique bi-invariant metric that gives unit length to S^1. We hope, in the future, to use our procedure for a compact Lie group G [5].

Let E and F be units vector fields on $\Sigma = X$ for the metrics g and h, respectively; we will denote by $\tilde{F}$ the induced unitary section on $\phi^*(TX)$. We also denote by a and b infinitely differentiable functions on S^1: in this case we obtain

$$\mathbb{D}_{(g,\phi)}(a\,1 \otimes \tilde{F} + b\,E^* \otimes \tilde{F}) = -E(b)1 \otimes \tilde{F} + E(a)E^* \otimes \tilde{F}. \tag{10}$$

In order to compute the determinant of the Dirac operator above, first we consider the case in which g is the usual metric u on the unit circumference. In this case the eigenvalues are the integers, each of them with multiplicity two. Next we consider $g = \lambda u$: the multiplicities for the associated Dirac operators remain equal to two, but the eigenvalues are multiplied by the square root of λ. The general case is reduced to that just considered by observing that, for an arbitrary metric g on the unit circumference, we can find an orientation preserving diffeomorphism f such that the pullback of g by f is of the form studied with $\lambda = \frac{l_g}{2\pi}$: here l_g denotes the length of the unit circumference with the metric g. In all these cases the η-invariant associated with the computations of the determinant of $\mathbb{D}_{(g,\phi)}$ is equal to unity [8]. After some computation using the ζ regularization procedure [9], and by regularizing the infinite sum of the $\mathrm{sgn}(\lambda)$, for the negative eigenvalues λ we obtain

$$\det(\gamma\mathbb{D}_{(g,\phi)}) = -l_g^2\gamma^{-2}. \tag{11}$$

Using the same procedure as above we obtain

$$\det(\beta\mathcal{H}^E) = l_g^2\beta^{-1}. \tag{12}$$

The variety of critical points stratifies according to the degree of the map ϕ. In degree zero we obtain just the constant maps: this is the main contribution to the partition function. After identifying them with X, the metric induced on X by the supermetric $\mathcal{G}$ becomes $l_g h$. The same happens with the n-instanton ϕ, these are identified with X through the map ϕ goes to $\phi(1)$.

We take into consideration, the following fact concerning integration over the orbit space of $\mathcal{M}$ under the action of the diffeomorphism group. (See in particular section 2 (pp 5295–6) and section 3 (pp 5297–8) of [6].)

$$\int_{\mathcal{M}/\mathcal{D}} F(\overline{g})\frac{\mathrm{FP}(\overline{g})}{\mathrm{vol}(K_{\overline{g}})}\nu_{\overline{\mathcal{G}}} = \int_0^\infty F(x)\,\frac{x}{2}\,x^{-3/2}\,\frac{2}{\sqrt{x}}\,\mathrm{d}x. \tag{13}$$

In the above expression $\overline{\mathcal{G}}$ is the metric on the quotient space induced by imposing that the canonical projection is an isometry. The $\frac{x}{2}$ originates from the Faddeev–Popov determinant; the $x^{-3/2}$ factor reflects the volume of the isometry group associated with a given metric g, with the volume originating from the metric induced on the diffeomorphism group of S^1 by g. The last factor in the above integral, $\frac{2}{\sqrt{x}}\,\mathrm{d}x$, corresponds under the natural identification $\overline{g} \mapsto l_g$ to the volume induced on the orbit space by $\overline{\mathcal{G}}$.

Finally after integration on the orbit space we obtain

$$-[Z_{\mathrm{R}}^{\mathrm{SC}}]_0 = \gamma^{-2}\beta^{1/2}\alpha^{-3/4}\tfrac{1}{2}\,\Gamma(\tfrac{3}{4}). \tag{14}$$

We proceed now to compute the contribution of the degree-n instanton, corresponding to maps ϕ that are critical points of E_{b}^g. These are characterized by the equation

$$\phi_*E = \frac{n}{l_g}\tilde{F}. \tag{15}$$

The bosonic energy is given by $E_{\mathrm{b}}^g(\phi) = \frac{n^2}{2l_g}$ and the topological energy E_{t} is precisely n.

After some substitutions in (9) we obtain for the n-instanton contribution

$$-[Z_{\mathrm{R}}^{\mathrm{SC}}]_n = q^n \gamma^{-2}\beta^{1/2}\alpha^{-3/4}\int_0^\infty \exp\left(-x^2 - \frac{bn^2}{2x}\right) x^{1/2}\,\mathrm{d}x. \tag{16}$$

In formula (16) $q = \exp(-\delta c)$ and $b = \beta\sqrt{\alpha}$. In this case (see (7)) we choose $B = cv_h$; note that v_h is the integer generator of $H^1_{\mathrm{DR}}(X)$.

The integral $I(b,n)$ in formula (16) can be calculated using Maple and gives an expression involving hypergeometric functions, here denoted by F (see (18) below). Note that $I(b,n) = I(b,-n)$ and $q^n + q^{-n} = 2ch(n\delta c)$; with the above notations we can write at least formally our final formula for

$$-Z_{\mathrm{R}}^{\mathrm{SC}} = \gamma^{-2}\beta^{1/2}\alpha^{-3/4}\left[I(b,0) + \sum_1^\infty I(b,n)2ch(n\delta c)\right] \tag{17}$$

with

$$\begin{aligned} I(b,n) = {} & -\frac{1}{2}\Gamma\left(\frac{3}{4}\right) F\left([\ ], \left[\frac{1}{2}, \frac{1}{4}\right], -p^2\right) + \frac{\pi p\sqrt{2}}{\Gamma(\frac{3}{4})} F\left([\ ], \left[\frac{3}{2}, \frac{3}{4}\right], -p^2\right) \\ & -\tfrac{8}{3}\sqrt{2\pi p^3}\,F([\ ], [\tfrac{5}{4}, \tfrac{7}{4}], -p^2) \end{aligned} \tag{18}$$

where $4p = bn^2$.

5. Conclusions

We have outlined the main ingredients for building a supersymmetric nonlinear model and we have shown explicitly that they are sufficient to obtain a finite answer for the partition function of our toy model. We have not imposed geometric conditions on either Σ or X; in the first manifolds it is convenient to impose a spin structure and on the second we require the existence of complex or quaternionic structures compatible with the given metric h on X. This and related topics are discussed in [4, 10, 11].

With dimensions greater than two we need to impose additional restrictions in order to successfully compute the renormalized partition function. For instance in dimension two, for a compact oriented manifold of genus greater than unity without boundary we consider only metrics of constant curvature equal to -1. The next step is to compute correlation functions and to relate these computations to physical or geometrical problems as in [10]. We have computed the instanton corrections for all degrees; we have obtained explicit expressions involving hypergeometric functions but we do not know if the series obtained is convergent.

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